Structural identification

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Derivations of recursive and long-run impact matrices, an incompatible mixed restriction, and the conversion from annualized growth to levels.

Based on the 2024 BSE course taught by Luca Gambetti and Konstantin Boss. I have revised the classroom code and added the accompanying explanations. The notes identify the papers, data, and assumptions used in each application. These are my course notes, not an official BSE publication. A classroom replication should not be read as a replication of every result in its underlying paper.

1 A recursive shock

Write $B = \begin{pmatrix} a & 0 \\ b & c \end{pmatrix}$. Multiplication gives

$$BB' = \begin{pmatrix} a^2 & ab \\ ab & b^2 + c^2 \end{pmatrix}.$$

The positive-diagonal convention yields $a = 2$, then $b = 2/a = 1$, and finally $c = \sqrt{5-1} = 2$. Hence $B = \begin{pmatrix} 2 & 0 \\ 1 & 2 \end{pmatrix}$. A unit second shock affects variable two by two units on impact. Multiplying its column by $.5/2 = .25$ gives $(0, .5)'$.

If that column replaces the original one without changing its variance, the implied covariance becomes $\begin{pmatrix} 4 & 2 \\ 2 & 1.25 \end{pmatrix}$, not the observed Σ. The normalized response is a different-sized experiment, not a new unit-variance decomposition of the same residual covariance. Keep the original factor for variance accounting.

2 The infinite-horizon zero

The exact multiplier is

$$C(1) = (I - A)^{-1} = \text{diag}(2, 4/3).$$

Consequently,

$$C(1)\Sigma C(1)' = \begin{pmatrix} 16 & 16/3 \\ 16/3 & 80/9 \end{pmatrix}.$$

Its lower-triangular factor is $D = \begin{pmatrix} 4 & 0 \\ 4/3 & 8/3 \end{pmatrix}$: the bottom-right covariance is $(4/3)^2 + (8/3)^2 = 80/9$. Premultiply by $I - A$:

$$B = \begin{pmatrix} 1/2 & 0 \\ 0 & 3/4 \end{pmatrix} \begin{pmatrix} 4 & 0 \\ 4/3 & 8/3 \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 1 & 2 \end{pmatrix}.$$

The direct check gives $C(1)B = \begin{pmatrix} 4 & 0 \\ 4/3 & 8/3 \end{pmatrix}$, whose upper-right element is zero. In this diagonal-dynamics example the short-run and long-run restrictions select the same factor. That is a feature of this fixture, not a general equivalence: cross-equation dynamics usually make the long-run impact matrix nontriangular.

3 Conflicting restrictions

Write $q_1 = (a, b)'$. The impact restriction gives $a = 0$. The second row of $C(1)$ gives the long-run restriction $a + 2b = 0$. Substituting $a = 0$ yields $b = 0$. The only solution is therefore the zero vector, which violates $q_1' q_1 = 1$.

The stacked restriction matrix $\begin{pmatrix} 1 & 0 \\ 1 & 2 \end{pmatrix}$ has rank two and no nontrivial null space. There is no admissible shock direction. More restrictions do not necessarily produce a better-identified model; they may produce an inconsistent model. A single impact restriction here would leave direction $(0, 1)'$ up to sign. The additional long-run restriction rules it out.

4 Annualized growth

Annualized quarterly log growth is four times quarterly log growth. The underlying quarterly responses are therefore 1, .5, and 0. Cumulating yields

$$L_0 = 1, \qquad L_1 = 1 + .5 = 1.5, \qquad L_{10} = 1.5.$$

The return of growth to zero means that the log-level difference stops increasing; it does not imply that the accumulated level difference vanishes. An offsetting negative growth response would be needed for that.

If the supplied 4 and 2 were already nonannualized quarterly growth, no division by four would be appropriate. The level responses would be 4, 6, and 6. The discrepancy is exactly a units error, even though both calculations use the same cumulative-sum command.