

External instruments

Tyler Sotomayor

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Derivations and hand calculations for proxy moments, first-stage equivalence, calendar matching, and structural shock normalization.

Based on the 2024 BSE course taught by Luca Gambetti and Konstantin Boss. I have revised the classroom code and added the accompanying explanations. The notes identify the papers, data, and assumptions used in each application. These are my course notes, not an official BSE publication. A classroom replication should not be read as a replication of every result in its underlying paper.

1 Recovering a direction

The policy covariance is $m_p = 0.4$, hence

$$s = m/0.4 = (-0.5, -0.25, 1, 0.3)'.$$

For a one-percentage-point increase in the policy indicator, the other impact responses are -0.5 , -0.25 , and 0.3 in their respective measurement units. They are not all automatically percentage points: the variable definitions determine that interpretation.

Multiplying the instrument by -100 gives $\tilde{m} = (20, 10, -40, -12)'$. Dividing by -40 produces exactly the same s . Both a change of units and a reversal of instrument orientation cancel in the ratio. The check is $s_p = 1$ in both calculations. This invariance does not say that replacing the instrument with a substantively different series leaves the direction unchanged.

2 The two-stage algebra

Let $a = z'u_p/(z'z)$, so $\hat{u}_p = az$. Provided $z'z > 0$ and $a \neq 0$,

$$\hat{\delta} = \frac{az'u_q}{a^2z'z} = \frac{z'u_q}{az'z} = \frac{z'u_q}{z'u_p}.$$

The fitted policy component isolates the variation associated with the instrument. Regressing each other innovation on that component recovers its impact relative to the policy variable. Using the policy innovation itself in the second stage would generally include variation driven by other structural shocks.

For an intercept, replace z , u_p , and u_q by their deviations from their means **over the same overlap sample**. Then apply the derivation to the centered vectors. An intercept in the full-sample VAR makes each innovation column sum to zero over the VAR sample; it does not make a shorter subsample sum to zero. This is why the centered and original no-intercept calculations can differ.

3 A missing month

The usable pairs are $(u_{\text{Jan}}, 1)$, $(u_{\text{Mar}}, 0)$, and $(u_{\text{Apr}}, -1)$. February is absent because its instrument is missing. March remains: its zero is observed and contributes to the sample mean, sample count, and first-stage residual variance, even though its uncentered product with the instrument is zero.

Pairing the first three residual rows with $(1, 0, -1)$ instead produces January, February, and March pairs. The final surprise is assigned to March rather than April, and February receives a fabricated observed zero. Equal vector lengths do not validate this merge.

In the revised implementation, `ismember(model.periods,zPeriods)` supplies the calendar mapping, and `isfinite` is applied after matching. The calendar-gap unit test deliberately includes an instrument-only month with a huge value; that value must never enter an identifying moment.

4 Unit variance and weak normalization

Here $s'\Sigma_u^{-1}s = 2^2/4 + 1^2/1 = 2$. Therefore

$$b = \frac{1}{\sqrt{2}}(2, 1)' = (\sqrt{2}, 1/\sqrt{2})'.$$

At impact, $C_0 = I$. Variable one's share is $b_1^2/4 = 2/4 = 1/2$. Variable two's share is $b_2^2/1 = (1/2)/1 = 1/2$. Using the policy-unit vector s instead would incorrectly give shares of one for both variables. The check $b'\Sigma_u^{-1}b = 1$ establishes the intended variance scale.

In the weak-normalization example, the first relative impact is initially $0.2/0.001 = 200$. Changing only the small denominator to 0.002 gives $0.2/0.002 = 100$. Both ratios are numerically defined, but the inferred response changes by half after a covariance change of only 0.001. This illustrates denominator sensitivity. A programming guard against division by zero is not a weak-instrument-robust confidence procedure, and the example itself is not a formal weak-identification test.